# Blackbody radiation with compact and non-compact extra dimensions

Ramaton Ramos* and Henrique Boschi-Filho†

*Instituto de Física, Universidade Federal do Rio de Janeiro,*

*Caixa Postal 68528, RJ 21941-972 – Brazil*

## Abstract

The problem of blackbody radiation in flat space with non-compact extra dimensions was analysed recently. In the present article we reanalyse this problem with compact (and non-compact) extra dimensions in flat space to observe the consequences of this approach upon Wien's displacement and Stefan-Boltzmann law.

---
*Electronic address: ramaton@gmail.com
†Electronic address: boschi@if.ufrj.br

## I.   INTRODUCTION

The common sense tell us and most physical theories assume that the space we live is three dimensional. Newton, in his *Principa*, assumed a three dimensional absolute space, besides an absolute time. The electromagnetic theory was developed on a three dimensional space with an independent time. Einstein revolutioned the concepts of space and time with the theory of relativity. In the special and general theories of relativity the space-time is supposed to be four dimensional. These theories have been tested and confirmed by experience, in a certain domain of validity.

Kaluza and Klein, in the 1920's proposed a model in a five dimensional space-time to unify gravity, described by the general theory of relativity, with Maxwell's electromagnetism. In this model, the fifth dimension is compactified on a circle of a given radius, while the other four non-compact dimensions are identified with our usual space-time.

String theory is presently the most viable candidate for unifying gravity with the others fundamental interactions, described by quantum field theories. String theory requires a higher dimensional 9+1 space-time [1, 2]. Usually, the six extra spatial dimensions are supposed to be compact, with a compactification parameter related to the Planck scale.

About a decade ago, different models have been proposed to deal with the compactification problem in string theory, keeping the extra dimensions compact [3, 4] or non-compact [5, 6]. Various predictions of these models will be tested in the forthcoming LHC experiment, with beams at the 15TeV scale.

Another way to test the predictions of string theory is to look at the physics of low energy processes. It is expected that, in some way, string theory could reproduce the physics we observe in our everyday life. Recently, the behaviour of the Wien's displacement and the Stefan-Boltzmann law in a space with non-compact extra dimensions was analysed [7, 8]. In these works, it was shown that the Wien's displacement and the Stefan-Boltzmann law are sensitive to the number of non-compact extra dimensions. Here, we reanalyse this problem considering compact and non-compact flat extra dimensions.

## II. CAVITY RADIATION

A blackbody can be defined as a body whose surface completely absorbs all radiation falling upon it. As a consequence, all blackbodies emit thermal radiation with the same spectrum. In order to study its single properties we consider a small orifice connecting an isothermal enclosure to its outside as its simplest example. Here we consider a blackbody immersed in a $D$-dimensional flat space and assume that the enclosure in question is a hyper-parallelepiped with $D$ dimensions.

The spectral energy density $\rho_T(\nu)$ is the energy contained in the interval of frequencies between $\nu$ and $\nu + d\nu$ per frequency and volume inside an isothermal enclosure maintained at temperature $T$. It is proportional to the spectral energy rate per unit area of the orifice, which must have the same behaviour of a blackbody since it absorbs almost completely all radiation falling upon it:

$$\rho_T(\nu) \propto R_T(\nu) \tag{1}$$

These are related by geometric factors and the speed of light, but the exact relation is irrelevant for our discussion.

Let us consider the electromagnetic radiation inside the enclosure as standing waves. Then, for each wave, we will deal with $D - 1$ possible directions of polarisation for the electric field. Choosing a system of orthogonal Cartesian coordinates with origin at one of the enclosure's vertices, whose edges with length $\ell_i$ are parallel to its axes $x_i$, we take the $i$-th component of the electric field as:

$$E_i(x_i, t) = E_0 \sin\left(\frac{2\pi x_i}{\lambda_i}\right) \sin(2\pi \nu t) ; \quad i = 1, 2, 3, ..., D. \tag{2}$$

We assume that the components of the electric field satisfy the Dirichlet boundary condition $E_i(\ell_i, t) = 0$ on the cavity walls, which implies

$$\frac{2\ell_i}{\lambda_i} = n_i \tag{3}$$

and $n_i = 1, 2, 3, ...$ represent the possible modes of vibration.

Note that if the space is non-compact (infinite) there are no restrictions on the size of the enclosure's edges. But if the space is compact (finite) the enclosure's edges must be minor or equal to the sizes of the corresponding dimensions.

Considering the standing waves as components of a plain electromagnetic wave (with wavelength $\lambda$), such that $\theta_i$ are the angles between the propagation direction and the axes

$x_i$, so $\lambda_i = \lambda/\cos\theta_i$. Substituting this relation in (3), taking the square and summing over all spatial dimensions one finds:

$$\left(\frac{2\nu}{c}\right)^2 \sum_{i=1}^{D} \ell_i^2 \cos^2\theta_i = \sum_{i=1}^{D} n_i^2 \,, \tag{4}$$

which is the equation of a hyper-ellipsoid.

Building a $D$-dimensional Cartesian net of points $P_{(n_1,n_2,...,n_D)}$ over all possible modes of vibration, in a way that the adimensional square distance from $P$ to the origin is given by $r^2 = \sum_{i=1}^{D} n_i^2$, in analogy with the three dimensional case [9, 10] Note that one can define a characteristic length $\ell_c$ such that $\sum_{i=1}^{D} \ell_i^2 \cos^2\theta_i = \ell_c^2$. Then, the above equation reduces to

$$\frac{2\ell_c}{c}\nu = r \,. \tag{5}$$

The number of points contained between two concentric spherical shells of radius $r$ and $r + dr$, $N(r)dr$, in one $D$-dimensional octant is

$$N(r)\, dr = \frac{1}{2^D}\, \Omega_{(D)}\, r^{D-1} dr \,, \tag{6}$$

once the point density in this space is equal to unity, and $\Omega_{(D)}$ is the hyper-solid angle in $D$-dimensional space. The number of standing waves between frequencies $\nu$ and $\nu + d\nu$ is then

$$N(\nu)d\nu = (D-1)\frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \left(\frac{\ell_c}{c}\right)^D \nu^{D-1} d\nu \,. \tag{7}$$

where we used that $\Omega_{(D)} = 2\pi^{D/2}/\Gamma(\frac{D}{2})$. Since the average energy obtained by Planck for each standing wave is $h\nu/(e^{\frac{h\nu}{kT}} - 1)$, the spectral energy density is

$$\rho_T(\nu)d\nu = \frac{2h(D-1)(\sqrt{\pi}\ell_c\nu)^D}{c^D\Gamma(\frac{D}{2})(e^{\frac{h\nu}{kT}} - 1)\left(\prod_{i=1}^{D} \ell_i\right)} d\nu \,. \tag{8}$$

Now, we can make an appropriate linear transformation over the Cartesian axes $\bar{x}_i = M_{ij}x_j$ given by the $D$-dimensional matrix

$$M = \begin{bmatrix} \frac{\ell_1}{\ell_c} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\ell_2}{\ell_c} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\ell_3}{\ell_c} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\ell_D}{\ell_c} \end{bmatrix} \,. \tag{9}$$

4

Imposing that this transformation preserves the volume, which implies that $\det M = 1$, one has

$$\ell_c{}^D = \prod_{i=1}^{D} \ell_i \, . \tag{10}$$

Using this result in eq. (8) we finally find that the spectral energy density of a blackbody in a $D$-dimensional compact space is given by

$$\rho_T(\nu) = \frac{2h(D-1)(\sqrt{\pi}\nu)^D}{c^D \Gamma(\frac{D}{2})(e^{\frac{h\nu}{kT}} - 1)} \, . \tag{11}$$

This outcome states that the spectral energy density and, consequently, the spectral radiance emitted through the enclosure's orifice are independent of its size or shape, as expected once the blackbody radiation is universal.

## III.  GENERALIZED WIEN'S DISPLACEMENT AND STEFAN-BOLTZMANN LAW

According to eq. (1), the maximum of the spectral radiance is obtained from the derivative of eq. (11) with respect to the frequency $\nu$:

$$z^{D-1}(e^z D - D - z e^z) = 0 \tag{12}$$

where $z = h\nu/kT$ , so that

$$e^z = \frac{D}{D - z} \, . \tag{13}$$

The above equation has a non-trivial solution $z = z_D$ for any value of $D$ ($D > 1$). Therefore,

$$z_D = \frac{h\nu_{max}}{kT} \tag{14}$$

where $\nu_{max}$ is the value of the frequency $\nu$ for which the spectral radiance is maximum. Then,

$$\nu_{max} \propto T. \tag{15}$$

which is the usual Wien's displacement, valid for any dimension. Note that the constant of proportionality, given by eq. (14), depends on $D$.

The total radiance emitted by a blackbody surface corresponds to the integral of $R_T(\nu)$ over all the frequencies $\nu$. Integrating eq. (11), we get

$$\rho_T = a_D \, T^{D+1} \, , \tag{16}$$

5

where the constant $a_D$ is given by

$$a_D = \frac{2(D-1)k^{D+1}}{\Gamma(\frac{D}{2})}\left(\frac{\sqrt{\pi}}{ch}\right)^D \zeta(D+1)\,\Gamma(D+1) \tag{17}$$

and $\zeta(x)$ is the zeta function defined by

$$\zeta(x) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{z^{x-1}}{e^z-1}dz\,. \tag{18}$$

Since $\rho_T \propto R_T$, one finds

$$R_T = \sigma_D T^{D+1} \tag{19}$$

which is the Stefan-Boltzmann law generalized for $D$ compact or non-compact spatial dimensions, where

$$\sigma_D = D(D-1)\frac{ck}{2\pi}\left(\frac{2\sqrt{\pi}k}{ch}\right)^D \Gamma(\frac{D}{2})\,\zeta(D+1)\,. \tag{20}$$

These results coincide with the ones obtained in [7, 8] considering non-compact extra dimensions.


## IV.  CONCLUSIONS

Like initially proposed, we derived Wien's displacement and Stefan-Boltzmann law assuming a D-dimensional compact (or non-compact) space. Our results indicate a dependence on the space dimensions which seems to be in contradiction with the experiments. In the Stefan-Boltzmann law the total radiance is proportional to the fourth power of the temperature, implying that in our analysis we might have $D = 3$.

Though experimental results are consistent with a 3-dimensional space, it is relevant to take notice that no special kind of curved dimension was taken into account here, which could significantly modify the above results.

The modern interpretation of our world from string theory point of view is that strings living in a (curved) ten dimensional space-time are dual to four dimensional gauge theories, which describe the fundamental interactions. This way, the string theory can be regarded as a mathematical description of our four dimensional reality, as suggested by Polyakov [11].

Anyway, the forthcoming LHC experiment will test the existence of extra dimensions among other predictions on the fundamental interactions.

---

[1] M. B. Green, J. H. Schwarz and E. Witten, *Cambridge, Uk: Univ. Pr. ( 1987) 469 P. (Cambridge Monographs On Mathematical Physics)*

[2] B. Zwiebach, "A first course in string theory," *Cambridge, UK: Univ. Pr. (2004) 558 p.*

[3] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **429**, 263 (1998) [arXiv:hep-ph/9803315].

[4] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **436**, 257 (1998) [arXiv:hep-ph/9804398].

[5] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [arXiv:hep-ph/9905221].

[6] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [arXiv:hep-th/9906064].

[7] T. R. Cardoso and A. S. de Castro, Rev. Bras. Ens. Fis. **27**, 559 (2005) [arXiv:quant-ph/0510002].

[8] H. Alnes, F. Ravndal and I. K. Wehus, J. Phys. A **40**, 14309 (2007) [arXiv:quant-ph/0506131].

[9] R. Eisberg and R. Resnick, Quantum Physics of Atoms, Molecules, Solids, Nuclei and Particles Wiley, New York, 1974.

[10] F. K. Richtmyer, E. H. Kennard and T. Lauritsen, Introduction to Modern Physics, 5th. ed. McGraw-Hill, New York, 1955.

[11] A. M. Polyakov, Int. J. Mod. Phys. A **14**, 645 (1999) [arXiv:hep-th/9809057].